JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 2, May 2017
http://dx.doi.org/10.14403/jcms.2017.30.2.239

FUZZY ALMOST q-CUBIC FUNCTIONAL EQATIONS

ChangIl Kim*

Abstract. In this paper, we approximate a fuzzy almost cubic function by a cubic function in a fuzzy sense. Indeed, we investigate solutions of the following cubic functional equation

$$\begin{aligned}&3f(kx+y)+3f(kx-y)-kf(x+2y)-2kf(x-y)\\&-3k(2k^2-1)f(x)+6kf(y)=0.\end{aligned}$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [24]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [1] for additive mappings, and by Rassias [22] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([5], [6], [7], [10], [20]).

Recently, the stability problems in the fuzzy spaces has been extensively studied ([13], [18], [19]). The concept of fuzzy norm on a linear space was introduced by Katsaras [15] in 1984. Later, Cheng and Mordeson [4] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. In

Received February 04, 2017; Accepted April 11, 2017.
2010 Mathematics Subject Classification: Primary 39B52, 46S40.
Key words and phrases: additive-quadratic mapping, fuzzy almost quadratic-additive mapping, fuzzy normed space.

2008, for the first time, Mirmostafaee and Moslehian ([18], [19]) used the definition of a fuzzy norm in [2] to obtain a fuzzy version of stability for the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called *a quadratic mapping.* In 2001, Rassias [23] introduced the following cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0 \tag{1.3}$$

and every solution of the cubic functional equation is called *a cubic mapping*

In this paper, we consider the following functional equation

$$\begin{aligned} &3f(kx+y) + 3f(kx-y) - kf(x+2y) - 2kf(x-y) \\ &- 3k(2k^2-1)f(x) + 6kf(y) = 0 \end{aligned} \tag{1.4}$$

for some fixed non-zero rational number k and show the generalized Hyers-Ulam stability of (1.4) in a fuzzy sense.

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,\ t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,\ t) = 1$ for all $t > 0$;
(N3) $N(cx,\ t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y,\ s+t) \geq \min\{N(x,s),\ N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X, N) and one denotes it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon > 0$, there is an $m \in N$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each

Cauchy sequence in it is convergent and the a complete fuzzy normed space is called *a fuzzy Banach space*.

2. Solutions and the stability of (1.4)

In this section, we investigate solutions of (1.4) and prove the generalized Hyers-Ulam stability of (1.4) in fuzzy Banach spaces.

We start with the following theorem.

THEOREM 2.1. *Let X and Y be normed spaces and $f : X \longrightarrow Y$ a mapping with $f(0) = 0$. Suppose that f satisfies (1.4) and $k \neq 0$. Then f is a cubic mapping.*

Proof. Suppose that f satisfies (1.4). If $k = 1$, then f satisfies (1.3) and so f is a cubic mapping. Suppose that $k \neq 1$.

Setting $y = 0$ in (1.4), we have

$$f(kx) = k^3 f(x) \tag{2.1}$$

for all $x \in X$ and setting $x = 0$ and $y = x$ in (1.4), we have

$$3f(x) + 3f(-x) = kf(2x) + 2kf(-x) - 6kf(x) \tag{2.2}$$

for all $x \in X$. Replacing x by $-x$ in (2.2), we have

$$3f(x) + 3f(-x) = kf(-2x) + 2kf(x) - 6kf(-x) \tag{2.3}$$

for all $x \in X$. Since $k \neq 0$, by (2.2) and (2.3), we have

$$f(2x) - f(-2x) = 8[f(x) - f(-x)] \tag{2.4}$$

for all $x \in X$. Relpacing y by ky in (1.4), by (2.1), we have

$$\begin{aligned} &3k^2[f(x + y) + f(x - y)] \\ &\quad - f(x + 2ky) - 2f(x - ky) - 3(2k^2 - 1)f(x) + 6k^3 f(y) = 0 \end{aligned} \tag{2.5}$$

for all $x, y \in X$ and letting $y = -y$ in (2.5), we have

$$\begin{aligned} &3k^2[f(x + y) + f(x - y)] \\ &\quad - f(x - 2ky) - 2f(x + ky) - 3(2k^2 - 1)f(x) + 6k^3 f(-y) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in X$. By (2.5) and (2.6), we have

$$[f(x+2ky)-f(x-2ky)]-2[f(x+ky)-f(x-ky)]-6k^3[f(y)-f(-y)] = 0 \tag{2.7}$$

for all $x, y \in X$. Letting $y = \frac{1}{k}y$ in (2.7), we have

$$[f(x+2y)-f(x-2y)]-2[f(x+y)-f(x-y)]-6[f(y)-f(-y)] = 0 \tag{2.8}$$

for all $x, y \in X$.

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_o satisfies (2.8). By (2.4), we have

$$f_o(2y) = 8f_o(y) \tag{2.9}$$

for all $y \in X$. Letting $x = 2x$ in (2.8), by (2.9), we have

$$4[f_o(x+y) - f_o(x-y)] = f_o(2x+y) - f_o(2x-y) + 6f_o(y) \tag{2.10}$$

for all $x, y \in X$. Interchanging x and y in (2.10), we have

$$4[f_o(x+y) + f_o(x-y)] = f_o(x+2y) + f_o(x-2y) + 6f_o(x) \tag{2.11}$$

for all $x, y \in X$. By (2.8) and (2.11), we have

$$f_o(x+2y) - 3f_o(x+y) + 3f_o(x) - f_o(x-y) - 6f_o(y) = 0$$

for all $x, y \in X$ and hence f_0 is a cubic mapping.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_e satisfies (2.8) and so we have

$$f_e(x+2y) - f_e(x-2y) - 2[f_e(x+y) - f_e(x-y)] = 0 \tag{2.12}$$

for all $x, y \in X$. Letting $y = x$ in (2.12), we have

$$f_e(3x) = 2f_e(2x) + f_e(x)$$

for all $x \in X$ and letting $y = 2x$ in (2.12), we have

$$f_e(4x) = 2f_e(3x) - 2f_e(x)$$

for all $x \in X$. Hence we have $f_e(4x) = 4f_e(2x)$ for all $x \in X$ and so

$$f_e(2x) = 4f_e(x), \ \ f_e(3x) = 9f_e(x), \ \ f_e(4x) = 16f_e(x)$$

for all $x \in X$. By induction on n, we have

$$f_e(nx) = n^2 f_e(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence

$$f_e(rx) = r^2 f_e(x) \tag{2.13}$$

for all $x \in X$ and all rational number r. Since k is a non-zero rational number, by (2.1) and (2.13), we have

$$k^3 f_e(x) = k^2 f_e(x).$$

Since $k \neq 0, 1$, $f_e(x) = 0$ for all $x \in X$. Hence $f = f_o + f_e = f_o$ is a cubic mapping. $\square$

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$\begin{aligned} Df(x,y) = {} & 3f(kx+y) + 3f(kx-y) - kf(x+2y) - 2kf(x-y) \\ & - 3k(2k^2-1)f(x) + 6kf(y) \end{aligned}$$

for all $x, y \in X$ and some non-zero real number k.
For a given real number q with $q > 0$, the mapping f is said to be *a fuzzy q-almost cubic mapping if*

$$N'(Df(x,y),\ t+s) \geq \min\{N(x,\ t^q), N(y,\ s^q)\} \tag{2.14}$$

for all $x, y \in X$ and all positive real numbers t, s.

THEOREM 2.2. *Let q be a positive real number withwith $|k|^{3q-1} > 1$ and $f : X \longrightarrow Y$ a fuzzy q-almost cubic mapping with $f(0) = 0$. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that*

$$N'(F(x) - f(x), t) \geq N\Big(x, \frac{3^q(|k|^3 - |k|^p)^q t^q}{2^q |k|^{3q-1}}\Big) \tag{2.15}$$

for all $x \in X$ and all $t > 0$, where $p = \frac{1}{q}$.

Proof. Letting $y = 0$ and $s = t$ in (2.14), we get

$$N'\Big(f(kx) - k^3 f(x), \frac{t}{3}\Big) \geq N(x, t^q) \tag{2.16}$$

for all $x \in X$ and all $t > 0$ and replacing x by $k^n x$ in (2.16), we get

$$N'\Big(f(k^{n+1}x) - k^3 f(k^n x), \frac{t}{3}\Big) \geq N\Big(x, \frac{t^q}{|k|^n}\Big)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$. Hence we have

$$N'\Big(f(k^{n+1}x) - k^3 f(k^n x), \frac{1}{3}|k|^{\frac{n}{q}} t^{\frac{1}{q}}\Big) \geq N(x, t)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$ and so we get

$$N'\Big(\frac{f(k^{n+1}x)}{k^{3(n+1)}} - \frac{f(k^n x)}{k^{3n}}, \frac{1}{3}|k|^{n(p-3)-3} t^p\Big) \geq N(x, t)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$. For $n > m \geq 0$, we have
(2.17)

$$\begin{aligned}
&N'\Big(\frac{f(k^n x)}{k^{3n}} - \frac{f(k^m x)}{k^{3m}}, \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big) \\
&= N'\Big(\sum_{i=m+1}^{n} \Big[\frac{f(k^i x)}{k^{3i}} - \frac{f(k^{i-1}x)}{k^{3(i-1)}}\Big], \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big) \\
&\geq \min\Big\{N'\Big(\frac{f(k^i x)}{k^{3i}} - \frac{f(k^{i-1}x)}{k^{3(i-1)}}, \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big)\ \Big|\ m+1 \leq i \leq n\Big\} \\
&\geq N(x, t)
\end{aligned}$$

for all $x \in X$ and all positive integer n.

Let $x \in X$, $c > 0$ and $\epsilon > 0$. By (N5), there is a t_0 such that

$$N(x, t_0) \geq 1 - \epsilon$$

for all $x \in X$. Since $|k|^{3q-1} > 1$, $|k|^{p-3} < 1$ and so $\sum_{n=1}^{\infty} |k|^{n(p-3)-3}\ t_0^p$ is convergent. Hence there is a positive integer n_0 such that for any $n \in \mathbb{N}$ with $n \geq n_0$, $\sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t_0^p < c$ and so for $n > m \geq 0$, we have

$$\begin{aligned}
&N'\Big(\frac{f(k^n x)}{k^{3n}} - \frac{f(k^m x)}{k^{3m}}, c\Big) \\
&\geq N'\Big(\frac{f(k^n x)}{k^{3n}} - \frac{f(k^m x)}{k^{3m}}, \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t_0^p\Big) \\
&\geq N(x, t_0) \geq 1 - \epsilon
\end{aligned}$$

and thus $\left\{\frac{f(k^n x)}{k^{3n}}\right\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, there is an $F(x)$ in Y such that

$$F(x) = N - \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}.$$

Clearly, $F : X \longrightarrow Y$ is a mapping. Letting $m = 0$ in (2.17), we get

$$N'\Big(\frac{f(k^n x)}{k^{3n}} - f(x), \sum_{i=1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big) \geq N(x, t)$$

for all $x \in X$ and all positive integer n and so we have

$$N'\Big(\frac{f(k^n x)}{k^{3n}} - f(x), t\Big) \geq N\Big(x, \frac{t^q}{\Big(\sum_{i=1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\Big)^q}\Big) \tag{2.18}$$

for all $x \in X$, $n \in \mathbb{N}$, and $t > 0$.

Now, we will show that F satisfies (1.4). Let $x, y \in X$. By (N4), we have

$$
\begin{aligned}
&N'(DF(x,y),t)\\
&\geq \min\Big\{N'\Big(3F(kx+y)-\frac{3f(k^n(kx+y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(3F(kx-y)-\frac{3f(k^n(kx-y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(kF(x+2y)-\frac{kf(k^n(x+2y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(2kF(x-y)-\frac{2kf(k^n(x-y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(3k(2k^2-1)F(x)-\frac{3k(2k^2-1)f(k^nx)}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(6kF(y)-\frac{6kf(k^ny)}{k^{3n}},\frac{t}{7}\Big),N'\Big(\frac{Df(k^nx,k^ny)}{k^{3n}},\frac{t}{7}\Big)\Big\}
\end{aligned}
$$

for all $x, y \in X$ and all positive integer n. The first six terms on the right-hand of the above inequality tend to 1 as $n \to \infty$ and by (2.14), we have

$$
N'\Big(\frac{Df(k^nx,k^ny)}{k^{3n}},\frac{t}{7}\Big) \geq N\Big(x,|k|^{(3q-1)n}\Big(\frac{t}{14}\Big)^q\Big)
$$

for all $x, y \in X$, all positive integer n and all $t > 0$. Since $|k|^{3q-1} > 1$, Hence $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.1, F is a cubic mapping.

Now, we will show that (2.15) holds. Let $x \in X$ and $t > 0$. By (2.18), for large enough n, we get

$$
\begin{aligned}
&N'(F(x)-f(x),t)\\
&\geq \min\Big\{N'\Big(F(x)-\frac{f(k^nx)}{k^{3n}},\frac{t}{2}\Big),N'\Big(f(x)-\frac{f(k^nx)}{k^{3n}},\frac{t}{2}\Big)\Big\}\\
&\geq N\Big(x,\frac{t^q}{\Big(\sum_{i=1}^{n}\frac{2}{3}|k|^{i(p-3)-3}\Big)^q}\Big)\\
&\geq N\Big(x,\frac{3^q(|k|^6-|k|^{p+3})^q t^q}{2^q|k|}\Big)
\end{aligned}
$$

for all $x \in X$ and all $n \in \mathbb{N}$ and so we have (2.15).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another cubic mapping satisfying (2.15). Then we have

$$N'(F(x) - F_1(x), t) = N'(F(k^n x) - F_1(k^n x), |k|^{3n} t)$$
$$\geq \min \Big\{ N'\Big(F(k^n x) - f(k^n x), |k|^{3n} \frac{t}{2}\Big), N'\Big(f(k^n x) - F_1(k^n x), |k|^{3n} \frac{t}{2}\Big) \Big\}$$
$$\geq N\Big(x, \frac{3^q(|k|^6 - |k|^{p+3})^q |k|^{n(3q-1)-1} t^q}{4^q}\Big)$$

for all $x, y \in X$, all positive integer n and all $0 < s < t$. Since $|k|^{3q-1} > 1$,

$$\lim_{n\to\infty} N\Big(x, \frac{3^q(|k|^6 - |k|^{p+3})^q |k|^{n(3q-1)-1} t^q}{4^q}\Big) = 1$$

and so $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$. □

THEOREM 2.3. *Let q be a positive real number withwith $|k|^{3q-1} < 1$ and $f : X \longrightarrow Y$ a fuzzy q-almost cubic mapping with $f(0) = 0$. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that*

$$N'(F(x) - f(x), t) \geq N\Big(x, \frac{3^q(|k|^p - |k|^3)^q t^q}{2^q |k|^{3q-1}}\Big) \tag{2.19}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = \frac{x}{k}$ in (2.16), we get

$$N'\Big(f(x) - k^3 f\Big(\frac{x}{k}\Big), \frac{t}{3}\Big) \geq N(x, |k| t^q) \tag{2.20}$$

for all $x \in X$ and all $t > 0$ and replacing x by $\frac{x}{k^n}$ in (2.20), we get

$$N'\Big(f\Big(\frac{x}{k^n}\Big) - k^3 f\Big(\frac{x}{k^{n+1}}\Big), \frac{t}{3}\Big) \geq N\Big(x, |k|^{n+1} t^q\Big)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$. Hence we have

$$N'\Big(k^{3n} f\Big(\frac{x}{k^n}\Big) - k^{3(n+1)} f\Big(\frac{x}{k^{n+1}}\Big), \frac{1}{3}|k|^{n(3-p)-p} t^p\Big) \geq N(x, t)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$. For $n > m \geq 0$, we have

(2.21)
$$N'\Big(k^{3n} f\Big(\frac{x}{k^n}\Big) - k^{3m} f\Big(\frac{x}{k^m}\Big) \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(3-p)-p}\ t^p\Big)$$
$$\geq \min \Big\{ N'\Big(k^{3i} f\Big(\frac{x}{k^i}\Big) - k^{3(i-1)} f\Big(\frac{x}{k^{i-1}}\Big), \frac{1}{3}|k|^{i(3-p)-p}\ t^p\Big) \ \Big|\ m+1 \leq i \leq n \Big\}$$
$$\geq N(x, t)$$

for all $x \in X$ and all positive integer n.

Let $x \in X$, $c > 0$ and $\epsilon > 0$. By (N5), there is a t_0 such that

$$N(x, t_0) \geq 1 - \epsilon$$

for all $x \in X$. Since $|k|^{3q-1} < 1$, $|k|^{3-p} < 1$ and so $\sum_{n=1}^{\infty} |k|^{n(3-p)-p}\ t_0^p$ is convergent. Hence there is a positive integer n_0 such that for any $n \in \mathbb{N}$ with $n \geq n_0$, $\sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(3-p)-p}\ t_0^p < c$ and so for $n > m \geq 0$, we have

$$\begin{aligned}
&N'\Big(k^{3n} f\Big(\frac{x}{k^n}\Big) - k^{3m} f\Big(\frac{x}{k^m}\Big), c\Big) \\
&\geq N'\Big(k^{3n} f\Big(\frac{x}{k^n}\Big) - k^{3m} f\Big(\frac{x}{k^m}\Big), \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(3-p)-p}\ t_0^p\Big) \geq N(x, t_0) \\
&\geq 1 - \epsilon
\end{aligned}$$

and thus $\Big\{k^{3n} f\Big(\frac{x}{k^n}\Big)\Big\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, there is an $F(x)$ in Y such that

$$F(x) = N - \lim_{n \to \infty} \frac{f(k^n x)}{k^{3n}}.$$

The rest of the proof is similar to Theorem 2.2. □

Using Theorem 2.2- 2.3, we have the following corollary :

Corollary 2.4. *Let q be a positive real number with $q \neq \frac{1}{3}$ and $f : X \longrightarrow Y$ a fuzzy q-almost cubic mapping. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that*

$$N'(F(x) - f(x), t) \geq \begin{cases} N\Big(x, \frac{3^q(|k|^3 - |k|^p)^q}{2^q |k|^{1-3q}} t^q\Big), & \text{if } |k|^{3q-1} > 1 \\ N\Big(x, \frac{3^q(|k|^p - |k|^3)^q}{2^q |k|^{3q-1}} t^q\Big), & \text{if } |k|^{3q-1} < 1 \end{cases}$$

for all $x \in X$ and all $t > 0$.

We can use Theorem 2.2-2.3 to get a classical result in the framework of normed spaces. For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x, t) = \begin{cases} 0, & \text{if } t < \|x\| \\ 1, & \text{if } t \geq \|x\| \end{cases}$$

a fuzzy norm on X. In [17], [18] *and* [19], *some examples are provided for the fuzzy norm N_X. Here using the fuzzy norm N_X, we have the following corollary.*

Corollary 2.5. *Let X and Y be normed spaces. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|Df(x,y)\| \le \|x\|^p + \|y\|^p \tag{2.22}$$

for a fixed positive number p such that $p \neq 3$. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that the inequality

$$\|F(x) - f(x)\| \le \begin{cases} \frac{2\|k\|^p}{3(\|k\|^6 - \|k\|^{p+3})}\|x\|^p, & \textit{if } |k|^{3q-1} > 1 \\ \frac{2\|k\|^3}{3(\|k\|^{2p} - \|k\|^{p+3})}\|x\|^p, & \textit{if } |k|^{3q-1} < 1 \end{cases}$$

holds for all $x \in X$.

Proof. By the definition of N_Y, we have

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & \text{if } s+t < \|Df(x,y)\| \\ 1, & \text{if } s+t \ge \|Df(x,y)\|. \end{cases}$$

for all $x, y \in X$ and all $s, t \in \mathbb{R}$. Now, we claim that

$$N_Y(Df(x,y), s+t) \ge \min\{N_X(x, s^q), N_X(y, t^q)\}$$

for all $x, y \in X$ and $s, t > 0$, where $q = \frac{1}{p}$. If $N_Y(Df(x,y), s+t) = 1$, then it is trivial. Suppose that $N_Y(Df(x,y), s+t) = 0$. Then $s + t < \|Df(x,y)\|$. If $s \ge ||x||^p$ and $t \ge \|y\|^p$, then, by (2.22),

$$\|Df(x,y)\| \le \|x\|^p + \|y\|^p < s + t,$$

which is a contradiction. Hence either $s < \|x\|^p$ or $t < \|y\|^p$, that is, either $N_X(x, s^q) = 0$ or $N_X(y, t^q) = 0$ and thus f is a fuzzy q-almost cubic mapping. By Theorem 2.2 and Theorem 2.3, we have the results. □

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2** (1950), 64-66.

[2] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces,* J. Fuzzy Math. **3** (2003), 687705.

[3] I. S. Chang and Y. H. Lee, *Additive and quadratic type of functional equation and its fuzzy stability,* Results Math. To Appear.

[4] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces,* Bull. Calcutta Math. Soc. **86** (1994), 429436.

[5] P. W. Cholewa, *Remarkes on the stability of functional equations,* Aequationes Math. **27** (1984), 76-86.

[6] K Cieplinski, *Applications of fixed point theorems to the hyers-ulam stability of functional equation-A survey,* Ann. Funct. Anal. **3** (2012), no. 1, 151-164.

[7] S. Czerwik, *On the stability of the quadratic mapping in normed spaces,* Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59-64.

[8] H. Drygas, *Quasi-inner products and their applications A. K. Gupta (ed.),* Advances in Multivariate Statistical Analysis, 13-30, Reidel Publ. Co., 1987.

[9] V. A. Faiziev and P. K. Sahoo, *On the stability of Drygas functional equation on groups,* Banach Journal of Mathematical Analysis, 01/2007; **1** (2007), 43-55.

[10] P. Găvruta, *A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings,* J. Math. Anal. Appl. **184** (1994), 431-436.

[11] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. **27** (1941), 222-224.

[12] C. I. Kim, G. Han, and S. A. Shim, *Hyers-Ulam Stability for a Class of Quadratic Functional Equations via a Typical Form,* Abs. and Appl. Anal. **2013** (2013), 1-8.

[13] H. M. Kim, J. M. Rassias, and J. Lee, *Fuzzy approximation of Euler-Lagrange quadratic mappings,* Journal of inequalities and Applications, **2013** (2013), 1-15.

[14] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces,* Kybernetica **11** (1975), 326334.

[15] A. K. Katsaras, *Fuzzy topological vector spaces II,* Fuzzy Sets Syst. **12** (1984), 143154.

[16] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces,* Kybernetica, **11** (1975), 326-334.

[17] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation,* Fuzzy Sets Syst. **159** (2008), 730738.

[18] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy almost quadratic functions,* Results Math. **52** (2008), 161177.

[19] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem,* Fuzzy Sets Syst. **159** (2008), 720729.

[20] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation,* Bulletin of the Brazilian Mathematical Society, **37** (2006), no. 3, 361376.

[21] A. Najati and M. B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces,* J. Math. Anal. Appl. **337** (2008), 399-415.

[22] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces,* Proc. Amer. Math. Sco. **72** (1978), 297-300.

[23] J. M. Rassias, *Solution of the Ulam stability problem for cubic mappings* Glasnik Matematički, **36** (2001), no. 56, 63-72.

[24] S. M. Ulam, *A collection of mathematical problems,* Interscience Publisher, New York, 1964.

*

Department of Mathematics Education
Dankook University
Yongin 16890, Republic of Korea
E-mail: `kci206@hanmail.net`